Market Selection with an Endogenous State

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Abstract

This paper explores market selection in general equilibrium when the state of the economy is endogenous. Analysis of consumer survival in this case requires solution of the model's dynamics, for which evolutionary game theory can be useful; for instance, if the state and beliefs are Markovian and utility logarithmic, then the dynamics of consumption shares are described by the replicator dynamics. This is illustrated in a simple exchange economy, and in a standard monetary economy with multiple long-run equilibria where a plausible form of inflation targeting serves to destabilize a liquidity trap in favor of the target equilibrium. *Journal of Economic Literature* Classification: C73, D84.

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1 Introduction

Sandroni (2000) and Blume and Easley (2006) offer a foundation for the "market selection hypothesis" (Alchian 1950; Friedman 1953; Cootner 1964; Fama 1965) that market forces will lead rational traders to flourish at the expense of irrational ones. They show that, if agents are equally patient and at least one has rational expectations, then a complete markets economy satisfying certain conditions will eventually be dominated by correct beliefs.[1] However, they take the true path of the economy as given, whereas many plausible models exhibit endogeneity in the economy's state. Allowing this feature complicates the standard analysis of consumer survival by requiring solution of the economy's dynamics. I show here that evolutionary game theory can be helpful in this respect; in particular, if utility is logarithmic and the economy's state and beliefs are Markovian in the beliefs' consumption shares, then market selection of those shares is described by the "replicator dynamics" (Taylor and Jonker 1978), with beliefs flourishing if and only if they outperform the economy's average belief in their ability to predict the evolving state. Whilst a Markovian state fits within the standard framework for market selection, assuming that beliefs are Markovian renders them endogenous, requiring them to be found in equilibrium; the notion of competitive equilibrium hence requires modification under this assumption, as in Dindo and Massari (2017).[2]

I stress that this is a fully rational model, with the replicator dynamics playing a descriptive rather than a behavioral role; there is no bounded rationality here other than the market selection literature's relaxation of rational expectations in favor of heterogeneous beliefs. Whilst this evolutionary game theoretic representation has thus been implicit in the market selection approach since the seminal contribution of Blume and Easley (1992), here I make it explicit. Blume and Easley (1992, p. 10) note that:

> "One might be tempted to apply the biological population processes that have found favor in evolutionary game theory [to market selection]. The implicit hypothesis would be that the population dynamic is the reduced form of a learning process or an adaptive process of strategy revision in a large population of players."

[1] There are important limits to this market selection; even in complete markets, when discount factors differ patience can compensate for the effect of bad forecasts by inducing higher savings. Kogan et al. (2006, 2017) provide conditions under which irrational traders survive in complete markets, and show that even if they vanish, they can still have an impact on asset prices. Massari (2017), meanwhile, provides a necessary and sufficient condition for a trader to vanish; see also Blume and Easley (2009). Agents with differing beliefs can survive in incomplete markets (Beker and Chattopadhyay 2010; Cogley, Sargent, and Tsyrennikov 2014; Cao 2018), as well as under learning (Beker and Espino 2011; Dindo and Massari 2017), endogenous investment rules (Bottazzi and Dindo 2014; Bottazzi, Dindo, and Giachini 2018), ambiguity aversion (Guerdjikova and Sciubba 2015), and recursive preferences (Borovicka 2020; Dindo 2019).

[2] Some market selection papers have placed particular conditions on beliefs (Jouini and Napp 2011; Branger, Schlag, and Wu 2015) and preferences (Muraviev 2013).

However, here I show that the replicator dynamics in fact arises naturally as a representation of belief dynamics under market selection, without the need for the usual evolutionary game theoretic assumptions on learning or adaptive behavior within a larger population.

Since we can straightforwardly find the solution trajectories of the replicator dynamics, we can use them to analyze the effects of market selection with an endogenous state, as I illustrate with a simple exchange economy in Subsection 4.1.[3] In macroeconomic general equilibrium models, it is common for the state of the economy to be endogenous, owing for instance to the role of agents' expectations in shaping macroeconomic variables. This can often lead to multiple equilibria, in which case it is not clear what "correct beliefs" the market might select for; in this case, the replicator dynamics can also offer a method of equilibrium selection. I illustrate this in Subsection 4.2 by first establishing equilibrium convergence under market selection in a standard Taylor rule model with iid beliefs and a plausible form of inflation targeting. I then add a zero lower bound and consequent liquidity trap to the economy, and show that the market selects the target equilibrium over the liquidity trap. I begin, however, in the next two sections by outlining the model and its evolutionary game theoretic representation.

2 Market Selection

Blume and Easley (2006) analyze an infinite horizon general equilibrium model in which consumers allocate their wealth across states of the world each period. Whereas a standard rational expectations model would equip a representative agent with correct beliefs about the evolving state of the world, they allow heterogeneity in beliefs, which then flourish or diminish according to the consumption success that they bring.

There is a finite space S of *states of the world*, with sequences of states in discrete time denoted $\sigma = (\sigma_0, \ldots) \in \Sigma$ and called *paths* of the economy. A state of the world here is in the Arrow–Debreu decision theoretic sense of a resolution of *ex ante* uncertainty, rather than the evolutionary game theoretic sense of "state" as a sufficient statistic for the evolution of the process. The "true" probability measure on the measurable space Σ (together with its product sigma-field) is denoted p. Let $\sigma^t = (\sigma_0, \ldots, \sigma_t)$ denote the partial history through date t of the path σ, Σ^t the set of such t-length partial histories, and $\mathscr{F}_t$ the product sigma-field of events measurable at date t. For any probability measure q on Σ, $q_t(\sigma) = q(\{\sigma_0 \times \cdots \times \sigma_t\} \times S \times S \times \cdots)$ is the (marginal) probability of the partial history σ^t, and $q_t(\sigma|\sigma^{t-1}) = q(\{\sigma_0 \times \cdots \times \sigma_t\} \times S \times S \times \cdots |\sigma^{t-1})$ is the (conditional) probability of the state σ_t following σ^{t-1}.

[3] There is a developed suite of software for diagrammatic analysis of such evolutionary dynamics, called *Dynamo*, available at https://www.ssc.wisc.edu/∼whs/dynamo/.

An economy contains I consumers, each with consumption set $\mathbb{R}_+$.[4] A *consumption plan* $c^i : \Sigma \to \prod_{t=0}^{\infty} \mathbb{R}_+$ is a sequence of $\mathscr{F}_t$-measurable $\mathbb{R}_+$-valued functions $\{c_t^i(\sigma)\}_{t=0}^{\infty}$. Consumer i's *endowment stream* is a particular known consumption plan denoted ω^i, his *beliefs* the probability measure p^i on Σ, and his utility function

$$U_i(c) = \mathrm{E}_{p^i} \left\{ \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i(\sigma)) \right\},$$

where $\beta_i \in (0,1)$ is a discount factor and $u^i : \mathbb{R}_+ \to [-\infty, \infty)$ is a payoff function on consumptions.[5] I will say that consumer i has *rational expectations* if $p^i = p$. Like Blume and Easley, I exploit the fact that, if $c^* = (c^{1*}, \ldots, c^{I*})$ is a Pareto-optimal allocation of resources, then there is a vector of welfare weights $(\lambda_1, \ldots, \lambda_I) \gg \mathbf{0}$ such that c^* solves the problem

$$\begin{aligned} \max_{(c^1,\ldots,c^I)} & \sum_i \lambda_i U^i(c), \\ \text{s.t.} \quad & \sum_i c^i - \omega \leq \mathbf{0} \\ & \forall t, \sigma, i \quad c_t^i(\sigma) \geq 0, \end{aligned} \tag{1}$$

where $\omega_t = \sum_i \omega_t^i$.[6]

The following assumptions are basic to the model:

Axiom 1 *There are logarithmic payoffs* $u^i(c^i) = \ln c^i$, $\forall i$.

Axiom 2 *We have* $\infty > \overline{F} = \sup_{t,\sigma} \sum_i \omega_t^i(\sigma) \geq \inf_{t,\sigma} \sum_i \omega_t^i(\sigma) = \underline{F} > 0$.

Axiom 3 *For all consumers i, all dates t, and all paths σ, if $p_t(\sigma) > 0$ then $p_t^i(\sigma) > 0$.*

Axioms 2 and 3 are two of Blume and Easley's axioms, whilst Axiom 1 is a special case of their more general setting, but one that is focal in asset pricing (Rubinstein 1976). For failures of market selection absent Blume and Easley's conditions, see Kogan et al. (2006).

(1) is a standard Negishi (1960) problem, the solution to which characterizes the consumption dynamics of a complete markets exchange economy in competitive equi-

[4]The model could be extended to allow for infinitely many consumers, but would then require the use of the infinite-dimensional replicator dynamics (Oechssler and Riedel 2001, 2002). A finite number of consumers is hence assumed for simplicity.

[5]Note that this utility function relies implicitly on the consumer's satisfaction of, e.g., the Savage (1954) axioms.

[6]Note that the first constraint here implies free disposal and non-storable endowments.

librium by the First Welfare Theorem.[7] Existence of competitive equilibrium is guaranteed under Axioms 1–3 by Peleg and Yaari (1970), and many standard models fit within this framework. For instance, in Sandroni's (2000) analysis of the Lucas (1978) "tree" model, the state of the world determines the value of the dividend produced by the trees, whilst a consumer's endowment stream is determined by his share of claims to those (non-storable) dividends.

3 Endogenous State

In the Blume and Easley (2006) model above, the true path of the economy was exogenously governed by the probability measure p. In this section, I allow the evolving state to be determined endogenously, thus complicating market selection analysis by requiring the solution of the resulting dynamics. Evolutionary game theory provides a useful tool in this context: If utility is logarithmic, and the state and beliefs are Markov in the economy's consumption shares, then the evolution of those consumption shares is described by Taylor and Jonker's (1978) replicator dynamics. In general, this captures the dynamics of a population whose strategies flourish if and only if they outperform the average payoff; here, it is beliefs that flourish if and only if they place a higher-than-average probability on the economy's realized state.

The first-order conditions for the solution to (1) are, for all σ and t, as follows:

1. there is a number $\theta_t(\sigma) > 0$ such that if $p_t^i(\sigma) > 0$, then

$$\lambda_i \beta_i^t u^{i'}(c_t^i(\sigma)) p_t^i(\sigma) - \theta_t(\sigma) = 0;$$

2. if $p_t^i(\sigma) = 0$, then $c_t^i(\sigma) = 0$.

I assume equal discount factors $\beta_i = \beta_j = \beta$, under which these conditions imply that, if i and j have not vanished, then

$$\frac{c_t^j(\sigma)}{c_t^i(\sigma)} = \frac{\lambda_j p_t^j(\sigma)}{\lambda_i p_t^i(\sigma)}.$$

Such an equation holds for each unordered pair of consumers, giving $I!/2(I-2)!$ equations, of which $I-1$ are independent, the solution to which has

$$x_t^i(\sigma) = \lambda_i \left(\frac{p_t^i(\sigma)}{\overline{p}_t(\sigma)} \right), \tag{2}$$

[7] See, for instance, Kehoe and Levine (1985, p. 437), Kehoe (1989, p. 368) or Kehoe, Levine, and Romer (1992, §2). Negishi aggregation of course only captures equilibrium behavior, and is likely inadequate for welfare analysis (see, e.g., Mas-Colell, Whinston, and Green 1995, §4.D), but it is quite sufficient for this paper's equilibrium characterization purposes.

where $x_t^i(\sigma) \equiv c_t^i(\sigma)/\sum_{j=1}^I c_t^j(\sigma)$ and $\overline{p}_t(\sigma) \equiv \sum_i \lambda_i p_t^i(\sigma)$ is the belief of the representative consumer under logarithmic utility (Rubinstein 1974, 1976; Jouini and Napp 2007). The *population profile* $x_t(\sigma) \equiv (x_t^1(\sigma), \ldots, x_t^I(\sigma))$ of consumption shares of the consumers in period t belongs to the $(I-1)$-dimensional simplex X.[8]

Then

$$\begin{aligned} x_t^i(\sigma) &= \lambda_i \frac{p_t^i(\sigma|\sigma^{t-1}) p_{t-1}^i(\sigma)}{\overline{p}_t(\sigma|\sigma^{t-1})\overline{p}_{t-1}(\sigma)} \\ &= \frac{p_t^i(\sigma|\sigma^{t-1})}{\overline{p}_t(\sigma|\sigma^{t-1})} x_{t-1}^i(\sigma). \end{aligned} \tag{3}$$

Intuitively, if a consumer's conditional belief on next period's state of the world gives higher probability to the realized state than does the weighted-average belief $\overline{p}$, then his consumption share grows.

Equation (3) already looks quite similar to the replicator dynamics (Taylor and Jonker 1978), but the weighted average here is timeless rather than evolving, and the beliefs condition on the entire partial history up to period $t-1$. However, the former issue is moot, as we can see by letting $\hat{p}_t(\sigma|\sigma^{t-1}) \equiv \sum_i x_{t-1}^i(\sigma) p_t^i(\sigma|\sigma^{t-1})$ be the evolving consumption-weighted-average conditional belief on the period-t state of the world:

Lemma 1 *For all t and σ, $\overline{p}_t(\sigma|\sigma^{t-1}) = \hat{p}_t(\sigma|\sigma^{t-1})$.*

The proof is immediate:

$$\begin{aligned} \overline{p}_t(\sigma|\sigma^{t-1}) &\equiv \frac{\overline{p}_t(\sigma)}{\overline{p}_{t-1}(\sigma)} \\ &= \frac{\sum_i \lambda_i p_t^i(\sigma|\sigma^{t-1}) p_{t-1}^i(\sigma)}{\overline{p}_{t-1}(\sigma)} \\ &= \sum_i x_{t-1}^i(\sigma) p_t^i(\sigma|\sigma^{t-1}) \equiv \hat{p}_t(\sigma|\sigma^{t-1}). \end{aligned}$$

Note that $\overline{p}_t(\sigma|\sigma^{t-1})$ need not equal $\sum_i \lambda_i p_t^i(\sigma|\sigma^{t-1})$; instead, we can see from the second line above that it is a weighted average of the $p_t^i(\sigma|\sigma^{t-1})$'s, but with weights of $\lambda_i p_{t-1}^i(\sigma)/\overline{p}_{t-1}(\sigma)$. That these weights are equal to the consumption shares $x_{t-1}(\sigma)$ is remarkable and not at all obvious *a priori*. With a numerical state space S, and letting $\overline{\mathrm{E}}_{t-1}\sigma_t \equiv \mathrm{E}_{\overline{p}}(\sigma_t|\sigma^{t-1})$ and $\widehat{\mathrm{E}}_{t-1}\sigma_t \equiv \mathrm{E}_{\hat{p}}(\sigma_t|\sigma^{t-1})$ be the expected period-t state under $\overline{p}_t(\sigma|\sigma^{t-1})$ and $\hat{p}_t(\sigma|\sigma^{t-1})$, respectively, it follows that $\widehat{\mathrm{E}}_{t-1} = \overline{\mathrm{E}}_{t-1}$.

I will say that there is a *Markovian state* if, for all $t > 0$ and σ, and conditional on the set of possible beliefs P, the state of the world $\sigma_t \in S$ is determined by a (possibly unknown) time-homogeneous function $T : (x_{t-1}(\sigma), \sigma_{t-1}) \mapsto \Delta(S)$, where $\Delta(S)$ is the

[8]This population profile corresponds to the usual evolutionary game theoretic notion of the "state" of the process *qua* a sufficient statistic for its evolution; in this paper, the "state" terminology is reserved for the realizations of uncertainty in the model.

set of probability measures on S. For instance, the state of the world σ_t might be a function of the representative consumer's expected next-period state $\overline{\mathrm{E}}_t\sigma_{t+1}$.[9] The path of the economy is thus endogenous under a Markovian state, which is novel in the market selection literature but poses no problems for the application of existing results, as discussed in Appendix A.

I will say that there are (time-homogeneous) *Markovian beliefs* $\rho^1, \ldots, \rho^I$ if, for each $p^i \in P \equiv \{p^1, \ldots, p^I\}$, all $t > 0$ and all σ, $p^i_t(\sigma|\sigma^{t-1}) = \rho^i(\sigma_t|x_{t-1}(\sigma), \sigma_{t-1}) > 0$. In words, consumer beliefs on next period's state of the world must depend only on the current consumption shares and state of the world, and each possible state must be believed to have positive probability. This assumption requires a redefinition of competitive equilibrium to allow for the resulting endogeneity of beliefs, as in Dindo and Massari (2017). Moreover, Markovian beliefs render the welfare weights of (1) endogenous in the manner of Kehoe, Levine, and Romer (1992), raising issues of equilibrium multiplicity, which are also discussed in Appendix A. The full support of each ρ^i on S is to guarantee the satisfaction of Axiom 3, and hence existence, but is of course stronger than necessary.

Proposition 1 *Suppose that Axioms 1–3 hold, that there is a Markovian state and beliefs, and that T and $\rho^1, \ldots, \rho^I$ are functions of the consumption shares $x_{t-1}(\sigma)$ alone. Then the population profile $x_t(\sigma)$ evolves according to the replicator dynamics.*

This is again immediate; if $s(x_{t-1}(\sigma)) \in S$ is the state realized in period t, (3) becomes

$$x^i_t(\sigma) = \frac{\rho^i(s(x_{t-1}(\sigma))|x_{t-1}(\sigma))}{\hat{\rho}(s(x_{t-1}(\sigma))|x_{t-1}(\sigma))} x^i_{t-1}(\sigma). \tag{4}$$

This equation defines the discrete-time replicator dynamics for a hypothetical *belief game* G, with:

- player set consisting of the economy's I consumers;
- common action set P, from which each player i "chooses" ρ^i; and
- payoffs of $\rho^i(s(x)|x)$, the probability that ρ^i places on the realized state.

To be clear, this game is not actually played, but is merely a useful device for characterizing the dynamics of (4). As in the standard replicator dynamics, player i's problem is like facing a single opponent with mixed strategy x, but unlike the usual random-matching setting, player i's expected payoffs need not be linear in x; we have a "playing

[9]The mapping T may or may not be founded on optimizing behaviour; for instance, government policy may affect the state of the world, and is often assumed to be an exogenous influence on macroeconomic models.

the field" game (Maynard Smith 1982) where an action's expected payoffs depend on some property of the whole population, here the state $s(\cdot)$. Since T need not be deterministic, $s(\cdot)$ may be a random variable and the resulting replicator dynamics could hence be stochastic (Foster and Young 1990; Fudenberg and Harris 1992; Cabrales 2000; Imhof 2005); in that case, let $(\tilde{x}_t)_{t=0,1,\ldots} = (\mathrm{E}_p\, x_t(\sigma))_{t=0,1,\ldots}$ denote the *expected motion* of consumption shares.

Remark 1 *In the literature on "optimal beliefs" (Brunnermeier and Parker 2005; Brunnermeier, Gollier, and Parker 2007), agents choose their beliefs to their best advantage, but in the absence of effects arising from the interaction of beliefs, and where payoffs are exogenous utilities. Strategic interaction in belief choice is explored by Jouini, Napp, and Viossat (2013), who analyze the consequences of the replicator dynamics operating on beliefs in a general equilibrium setting, again with exogenous utilities. They assume that beliefs flourish if they outperform average beliefs in the sense of giving higher expected utility in the resulting Walrasian equilibrium. This is reasonable if beliefs are determined by evolution and utilities coincide with biological fitnesses, or if agents imitate others' beliefs (Björnerstedt and Weibull 1996; Schlag 1998) or choose myopically optimal beliefs (Hofbauer, Sorin, and Viossat 2009). Here, by contrast, I consider the case where beliefs are determined neither by evolution nor by choice, but instead are selected by the market. I have derived rather than assumed the replicator dynamics as a description of the market selection of beliefs, and the relevant game on which it acts is not one of utilities or biological fitnesses, but the probabilities that the beliefs assign to the realized path of the economy.*

What is the importance of the assumptions here? The Markovian structure gives a Markovian evolutionary dynamic and time-homogeneous payoffs, whilst the assumed logarithmic utility and dependence of T and beliefs only on $x_{t-1}(\sigma)$ give the precise form of the replicator dynamics.[10] Heterogeneous logarithmic payoff functions, $\{u^i(c^i) = a_i \ln c^i\}_i$, would give a rescaled replicator dynamics, and it is possible to show conditions under which other well-known evolutionary dynamics—including the best-response dynamic, fictitious play and reinforcement learning—describe market selection in a similar fashion.

[10] Whilst the "population" of beliefs is finite, the space of consumption shares is uncountable, as under the replicator dynamics.

4 Examples

4.1 A simple exchange economy

Suppose that we have two states $\{0,1\}$ and two consumers, each endowed with a unit of consumption in each state and each period. The first consumer believes that state 0 occurs with probability 0.6 each period, whilst the second consumer believes that state 1 occurs with probability 0.6 each period, in each case independently of the previous state and consumption shares. Suppose further that, unbeknownst to the consumers, $T(x_{t-1}(\sigma))$ gives state 0 with probability 0.6 if $\overline{\mathrm{E}}_{t-1}(\sigma_t) \leq 1/2$ and state 1 with probability 0.6 otherwise, for all $t > 0$ and σ.

Blume and Easley's (2006) long-run survival results rely on the exogenously given true p, but here we must solve for the endogenous state dynamics. Since the conditions of Proposition 1 hold, we can use the replicator dynamics to track the consumption shares, and hence the path of the economy. This gives the stochastic process

$$(x_t^1, \sigma_t) = \begin{cases} \left.\begin{array}{ll} \left(\frac{0.6}{0.6x_{t-1}+0.4(1-x_{t-1})}x_{t-1}^1, 0\right) & w.p.\ 0.6 \\ \left(\frac{0.4}{0.4x_{t-1}+0.6(1-x_{t-1})}x_{t-1}^1, 1\right) & w.p.\ 0.4 \end{array}\right\} & \text{if } x_{t-1}^1 \geq 0.5 \\ \left.\begin{array}{ll} \left(\frac{0.6}{0.6x_{t-1}+0.4(1-x_{t-1})}x_{t-1}^1, 0\right) & w.p.\ 0.4 \\ \left(\frac{0.4}{0.4x_{t-1}+0.6(1-x_{t-1})}x_{t-1}^1, 1\right) & w.p.\ 0.6 \end{array}\right\} & \text{otherwise} \end{cases}.$$

For instance, starting in state 0, and since the welfare weights (λ_1, λ_2) under logarithmic utility are the initial wealth shares $(0.5, 0.5)$, period-0 consumption shares are $(0.6, 0.4)$ by (2); Figure 1 then charts an illustrative first 200 periods of the process $(x_t^1(\sigma))_{t=0,1,\ldots}$. We can see that, after a period of volatility, the first consumer dominates and the second consumer vanishes. Whilst the reverse selection may of course occur for particular path realizations, survival of the first consumer will be the average selection (under $\tilde{x}_t^i(\sigma)$) for these parameters, since the expected increments to $x_{t-1}^1 \in [0.5, 1)$ can be shown to be strictly positive. The selection becomes stronger, for instance, as T focuses more weight on the state closest to $\overline{\mathrm{E}}_{t-1}(\sigma_t)$. Of course, the reverse selection would apply if the process began in state 1.

4.2 A standard monetary economy

To take a macroeconomic example, let r_t be the real rate of interest, R_t the nominal rate of interest, and π_{t+1}^e the expected period-$(t+1)$ rate of inflation. Consider the model consisting of a Fisher equation,

$$R_t = r_t + \pi_{t+1}^e, \qquad \text{(F)}$$

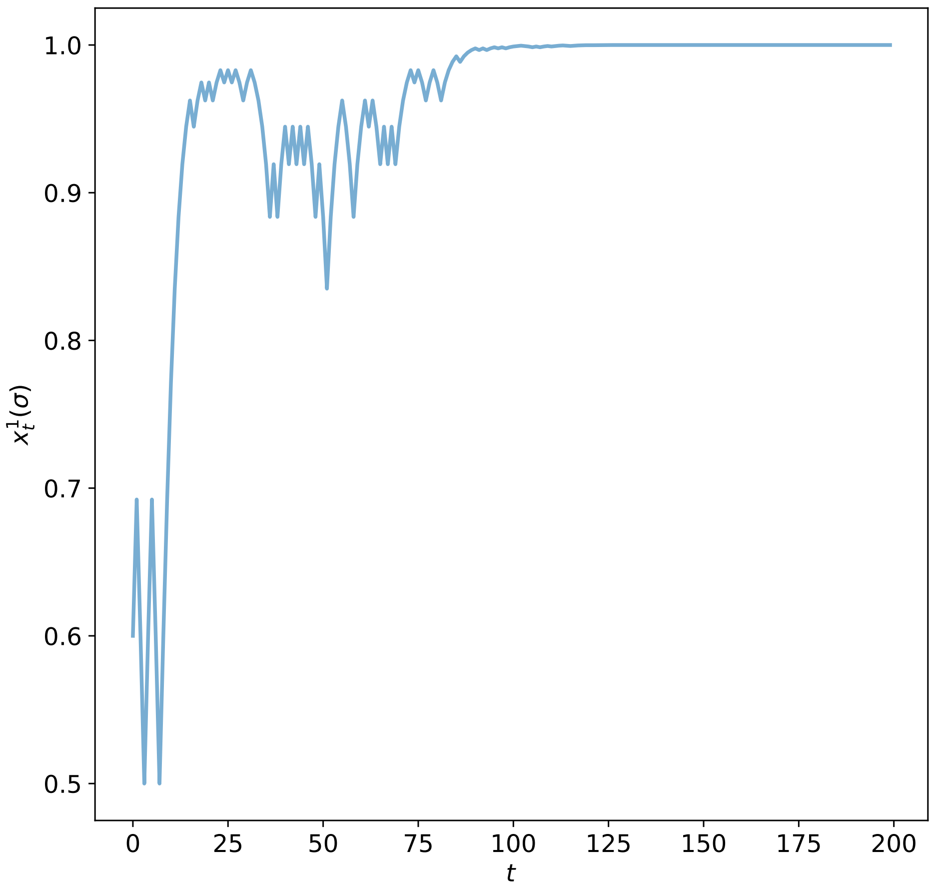

Figure 1: Consumption share $x_t^1(\sigma)$, $t = 0, 1, \ldots, 200$, given $\sigma_0 = 0$

and a Taylor rule with a target rate of inflation π_t^*,

$$R_t = \pi_t^* + r_t + \frac{1}{a}(\pi_t - \pi_t^*). \qquad \text{(TR)}$$

This is the frictionless limit of the New Keynesian model, which can be derived in the general equilibrium setting of Sections 2 and 3, as I do in Appendix B.

Inflation in the model is

$$\pi_t = \pi_t^* + a(\pi_{t+1}^e - \pi_t^*). \qquad (5)$$

In a stationary (or "locally bounded") equilibrium, $\pi_{t+1}^e = \pi_t$; under a constant inflation target $\pi_t^* = \pi^*, \forall t$, there is a unique such equilibrium $\pi_t = \pi^*, \forall t$, illustrated in Figure 2.[11] In this equilibrium, the economy follows a constant path on π^* for sure.[12]

[11]The locally bounded solution is also the "minimum state variable" (MSV) solution (McCallum 1981, 2003).

[12]However, there are many other equilibria satisfying $\pi_{t+1} = \pi^* + (1/a)(\pi_t - \pi^*)$ (implying an explosive path for inflation), and hence we have the classic indeterminacy of inflation under interest rate targeting (Sargent and Wallace 1975; Cochrane 2011).

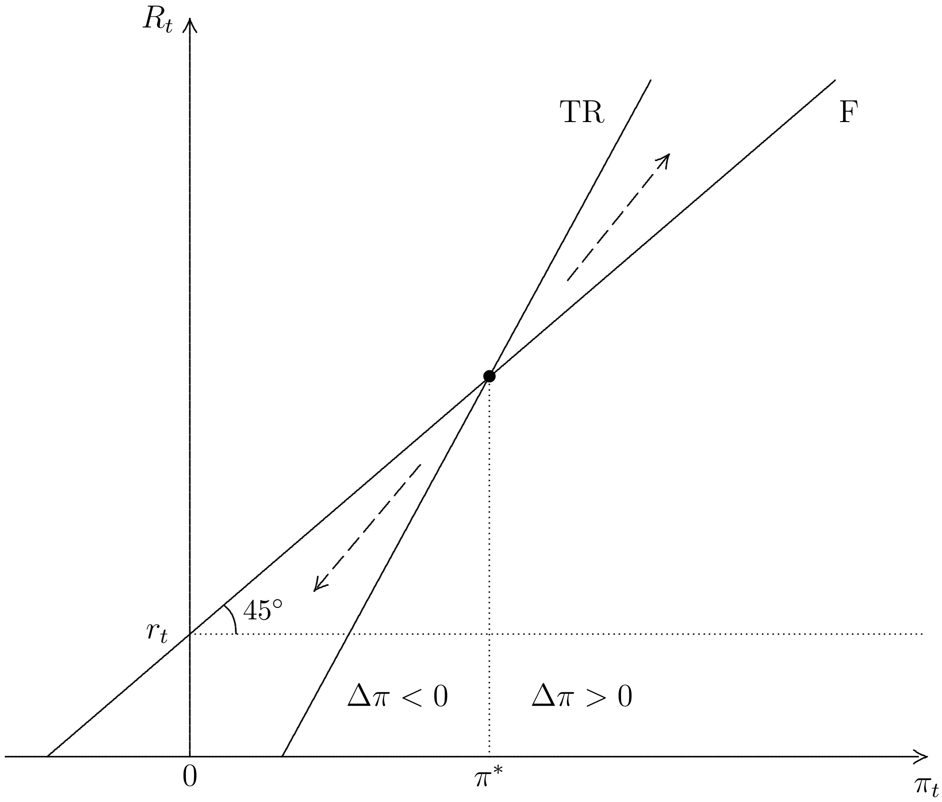

Figure 2: A Taylor rule

What will belief dynamics look like under market selection in this economy? I take the state of the world σ_t in Section 2 to be the inflation target π_t^* here, and I suppose that it takes values on a finite grid $S^\varepsilon = \{k, k+\varepsilon, k+2\varepsilon, \ldots, K-\varepsilon, K\} \subset \mathbb{Q}$. This could be due to a particular measurement accuracy for inflation, with the lower and upper bounds explained by the infeasibility of targeting inflation at very high (absolute) rates. Let an ε-*Dirac measure* be a probability measure on Σ that, for all t and all σ^{t-1}, places probability $(1-\varepsilon)$ on a particular $\tilde{s} \in S^\varepsilon$ in period t, and probability $\varepsilon/(|S^\varepsilon|-1)$ on each $s \in S^\varepsilon \backslash \{\tilde{s}\}$. Clearly such a measure constitutes a Markovian belief, and if all consumers hold such a belief, the weighted-average expectation $\overline{\mathrm{E}}_{t-1} \equiv \mathrm{E}_{\overline{p}}(\cdot|\sigma^{t-1})$ is independent of σ^{t-1}. Then, let $\hat{s} = \psi\overline{\mathrm{E}}_{t-1}\pi_t^* + (1-\psi)\pi^*$ for some $\psi \in [0,1)$ and $\pi^* \in S^\varepsilon$, with $\hat{s}$ lying in the real interval bounded by consecutive members $(\underline{s}, \overline{s})$ of S^ε, and let the Markovian state function $T(x_{t-1}(\sigma))$ assign probability $(\overline{s}-\hat{s})/(\overline{s}-\underline{s})$ to $\underline{s}$ and probability $(\hat{s}-\underline{s})/(\overline{s}-\underline{s})$ to $\overline{s}$ for all $t > 0$ and σ.[13] Thus, the policymaker targets some weighted average of a constant π^* and the period-t inflation target expected by the

[13]Recall that the conditional belief $\overline{p}_t(\sigma|\sigma^{t-1})$ is the same as the consumption-weighted conditional belief $\hat{p}_t(\sigma|\sigma^{t-1})$ by Lemma 1 above, and hence that $\overline{\mathrm{E}}_{t-1} = \widehat{\mathrm{E}}_{t-1}$, with $x_{t-1}(\sigma)$ sufficient to calculate $\overline{\mathrm{E}}_{t-1}$ under ε-Dirac measures.

representative consumer, with the stochasticity of T capturing (say) the measurement error associated with rounding inflation to a value in S^ε.[14] Under such a T, I say that the model has *populist inflation targeting*, noting that this nests the case $\psi = 0$ where the inflation target is an exogenous constant. A monetary policy shock (of which populist inflation targeting is an instance) is the standard choice of state for this model, and endogenizing it is also a common exercise (see, e.g., Cochrane 2011 p. 571).

Proposition 2 *Suppose that P is the set of all ε-Dirac measures, that there is populist inflation targeting, and that $\sigma_0 \in S^\varepsilon$ is given. Then the expected motion $(\tilde{x}_t)_{t=0,1,\ldots}$ of consumption shares approaches a Dirac measure on the equilibrium $\pi_t = \pi^*, \forall t$, as $t \to \infty$ and $\varepsilon \to 0$.*

Proof. With P the set of all ε-Dirac measures, the conditions of Proposition 1 are satisfied, and we will have $x_0^i(\sigma) > 0$ for all i under Axiom 2. X is a compact set that is positively invariant with respect to the autonomous system $(\tilde{x}_t)_{t=0,1,\ldots}$.[15] Moreover, (4) is locally Lipschitz in some open neighborhood Y of X in $\mathbb{R}^I$, by Lipschitz continuity of each $p_t(\sigma|\sigma^{t-1})$ in $x_{t-1}(\sigma)$ under T; hence, so is $\mathrm{E}_p\, x_t(\sigma)$.[16] For any t and σ, if i's belief is the ε-Dirac measure on the realized σ_t, then $x_t^i(\sigma) \to 1$ as $\varepsilon \to 0$ by (4), and hence $\overline{\mathrm{E}}_t \pi_{t+1}^* \to \sigma_t$ by Lemma 1. And since $\sigma_t \to \psi \overline{\mathrm{E}}_{t-1}\pi_t^* + (1-\psi)\pi^*$ in the same limit (as the approximation of the finite grid S^ε becomes exact), it follows that

$$\left|\overline{\mathrm{E}}_t \pi_{t+1}^* - \pi^*\right| \to \psi \left|\overline{\mathrm{E}}_{t-1}\pi_t^* - \pi^*\right| < \left|\overline{\mathrm{E}}_{t-1}\pi_t^* - \pi^*\right|$$

for $\sigma_t \neq \pi^*$ as $\varepsilon \to 0$. Hence, $V(\tilde{x}_t) \equiv \mathrm{E}_p \left|\overline{\mathrm{E}}_t \pi_{t+1}^* - \pi^*\right|$ is a continuous function such that $V(\tilde{x}_t) - V(\tilde{x}_{t-1})$ is nonpositive in X as $\varepsilon \to 0$, equalling 0 only at the point $\{x^*\}$ where the consumption share of the ε-Dirac measure on π^* is 1, which is an invariant set. Hence, the economy approaches the required equilibrium as $t \to \infty$ and $\varepsilon \to 0$ by LaSalle's Theorem (Bof, Carli, and Schenato 2018, Theorem 2.4). ■

Destabilizing a liquidity trap To this point, the economy has had a unique stationary equilibrium. Here, I show how modelling market selection via evolutionary dynamics can select between multiple stationary equilibria.

[14] Recall the existence of a representative trader with beliefs $\overline{p}$ under logarithmic utility (Rubinstein 1974, 1976).

[15] A set A is *invariant* with respect to $(\tilde{x}_t)_{t=0,1,\ldots}$ if $\tilde{x}_0 \in A \Rightarrow \tilde{x}_t \in A$, $\forall t \in \mathbb{R}$. It is *positively invariant* if $\tilde{x}_0 \in A \Rightarrow \tilde{x}_t \in A$, $\forall t \geq 0$.

[16] An autonomous system $x_t = f(x_{t-1})$ is *locally Lipschitz* in Y if each point in Y has a neighborhood Y_0 and a Lipschitz constant L_0 such that, for every $x, y \in Y_0$,

$$\|f(x) - f(y)\| \leq L_0 \|x - y\|.$$

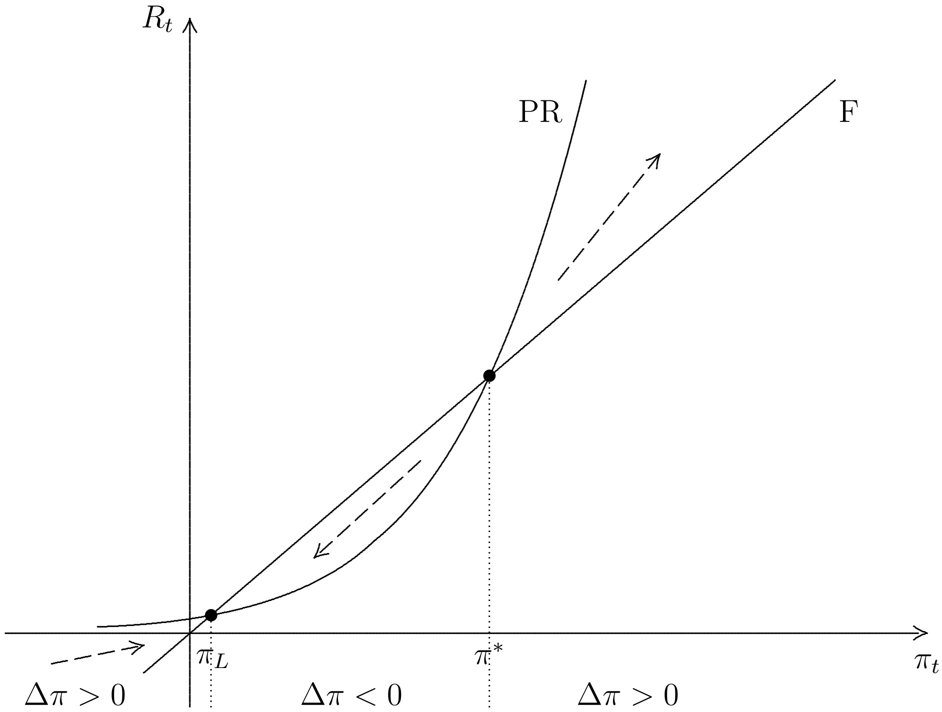

Figure 3: A liquidity trap

If we now posit a zero real interest rate and a zero lower bound to the nominal interest rate R_t, then Benhabib, Schmitt-Grohé, and Uribe (2001, 2002) show that adherence to the Taylor Principle (that the slope of the Taylor rule should exceed 1) in the vicinity of π^* implies a nonlinear policy rule $R_t(\pi)$ with a second locally bounded perfect foresight equilibrium $\pi_t = \pi_L, \forall t$ (local to which the Taylor Principle is violated), as illustrated in Figure 3. Specifically, I follow Benhabib, Schmitt-Grohé, and Uribe in analyzing the policy rule

$$R_t(\pi) = R_t^* e^{(A/R_t^*)(\pi_t - \pi_t^*)}, \qquad \text{(PR)}$$

where R_t^*, A and π^* are positive constants. In combination with the Fisher equation (F), this implies that the nominal interest rate is strictly positive and strictly increasing in the inflation rate, and that

$$\pi_t = \pi_t^* + \frac{R_t^*}{A} \ln\left(\frac{\pi_{t+1}^e}{R_t^*}\right). \qquad (6)$$

If there is a constant inflation target $\pi_t^* = \pi^*, \forall t$, then $R_t^* = \pi^*$ in equilibrium but $\pi_{t+1} = \pi^* e^{(A/\pi^*)(\pi_t - \pi^*)}$ is satisfied by constant paths on either π^* or π_L.

We are led to wonder which equilibrium to select.

Proposition 3 *Suppose that* $R_t^* = \pi^*, \forall t$*, that* P *is the set of all* ε*-Dirac measures, that there is populist inflation targeting, and that* $\sigma_0 \in S^\varepsilon$ *is given. Then the expected motion* $(\tilde{x}_t)_{t=0,1,\ldots}$ *of consumption shares approaches a Dirac measure on the equilibrium* $\pi_t = \pi^*, \forall t$*, as* $t \to \infty$ *and* $\varepsilon \to 0$*.*

Thus, under populist inflation targeting, the equilibrium $\pi_t = \pi^*, \forall t$, is selected over the liquidity trap, $\pi_t = \pi_L, \forall t$, with the proof identical to that of Proposition 2.

Appendices

A Markovian state and beliefs

Imposing a Markovian state and beliefs complicates the Blume and Easley (2006) model; the Negishi problem (1) is no longer a complete description of equilibrium. We can, however, employ the Kehoe, Levine, and Romer (1992) approach of characterizing equilibrium as the solution to a Negishi-style Pareto problem with side conditions.

Describing the standard Negishi problem (1) in the language of Kehoe, Levine, and Romer, the vector c is chosen to maximize the distorted social welfare function $\sum_i \lambda_i U^i(c)$, given a vector of parameters $(\lambda_1, \ldots, \lambda_I)$, a constraint set

$$\left\{ c : \sum_i c^i - \omega \leq \mathbf{0} \text{ and } \forall i, t, \sigma,\, c_t^i(\sigma) \geq 0 \right\},$$

and the additional conditions that Negishi's savings functions be zero (giving binding budget constraints). It can be shown that the welfare weights do not depend on the realized path of the economy. How can this be? Surely the realized path could determine whether consumer i vanishes, and hence whether he receives positive weight in the Pareto problem? No, the fault in the logic here is that consumer i could vanish and still receive positive expected utility under his (incorrect) belief p^i. Since the Negishi problem makes no reference to p, its solution remains unchanged and indeed Pareto optimal under a Markovian state; competitive equilibrium is unaffected by endogenizing the economy's path, and Blume and Easley's results continue to apply given a true path of play. This is a particularly simple case of the Kehoe, Levine, and Romer framework where the parameters of the original Negishi problem are unaltered (in contrast to their examples with endogenous parameters).

Matters are more complicated under Markovian beliefs $\rho^1, \ldots, \rho^I$, since $p^1, \ldots, p^I$ are now endogenously determined, necessitating a redefinition of competitive equilibrium. In particular, if σ_t and $x_t(\sigma)$ are $\mathscr{F}_t$-measurable, each belief p^i must solve the following infinite-dimensional fixed point problem:

$$p_t^i(\sigma) = \int_\Sigma \rho^i(\sigma_t|x_{t-1}(\sigma), \sigma_{t-1})\, dp_{t-1}^i(\sigma), \quad t = 1, \ldots, \qquad p_0^i(\sigma) = \rho_0^i, \tag{7}$$

where $\rho_0^i \in \Delta(S)$. For any fixed point p^i of this problem, the first-order conditions of the Pareto problem (1) must hold for the economy to be in equilibrium; I call c a *competitive equilibrium with Markovian beliefs* $\rho^1, \ldots, \rho^I$ if c solves (1) under $p^1, \ldots, p^I$ solving (7). There is a substantial macroeconomic literature on the existence of Markov (or recursive) equilibrium in competitive market economies. This is straightforward in the standard frictionless, representative consumer case, and indeed under additional side conditions (capturing frictions or endogenous variables) over a finite horizon, but in infinite discrete time existence is an open question (Santos 2002; Kubler and Schmedders 2002).[17]

Hence, we must verify existence of a competitive equilibrium with Markovian beliefs: Given $\rho_0^1, \ldots, \rho_0^I$ and $\lambda_1, \ldots, \lambda_I$, $x_0^i(\sigma) = \lambda_i \left(\frac{p_0^i(\sigma)}{\overline{p}_0(\sigma)}\right)$ from (2); then given $\rho^1, \ldots, \rho^I$ and σ_0, $p_1^i(\sigma) = \int_\Sigma \rho^i(\sigma_1|x_0(\sigma), \sigma_0)\, d\rho_0^i(\sigma_0)$, and $x_1^i(\sigma) = \lambda_i \left(\frac{p_1^i(\sigma)}{\overline{p}_1(\sigma)}\right)$. Iterating forwards, we have a consistent set of finite-dimensional distributions, and hence unique beliefs $p^1, \ldots, p^I$ on Σ by the Kolmogorov Extension Theorem. Since Markovian beliefs guarantee that Axioms 1–3 hold, and in particular that consumption is valued along all paths by all consumers, a competitive equilibrium with Markovian beliefs then exists by Peleg and Yaari (1970). But the welfare weights that solve (1) will depend on the consumers' beliefs, and the parameters of the optimization problem are hence endogenous, so that there may be multiple such equilibria. Indeed, there can be a robust continuum of equilibria near a steady state, i.e. indeterminacy (see Kehoe, Levine, and Romer 1992, §5). However, under each fixed point we have a Pareto problem (1), whose Euler equations then interact with the fixed point to determine the dynamics of that equilibrium.

There may be multiple equilibria even in static economies, as in the heterogeneous quasilinear utility examples of Shapley and Shubik (1977) and Bergstrom, Shimomura, and Yamato (2009), but they are endemic in dynamic models under the imposition of side conditions (Kehoe, Levine, and Romer 1992), and there have been many attempts to select between them.[18] In infinite horizon models, a popular equilibrium selection criterion is "stability" (Blanchard and Kahn 1980; Obstfeld and Rogoff 1986), in the sense

[17] Hellwig (1982) provides an incomplete markets example in which the unique rational expectations equilibrium is not Markov.

[18] For a review, see Driskill (2006).

that the model's variables converge to a stationary state, often owing to transversality conditions.[19] But some models have multiple stable equilibria (Calvo 1978), others have none that are stable (McCallum 1999, pp. 625–6), and others still have nominal explosions that transversality conditions cannot rule out (Cochrane 2011). An alternative criterion requires that a solution be derived under the assumption that agents' expectations are a function of a minimal set of state variables (Wallace 1980; McCallum 1983), though the basis for this assumption and its resultant exclusion of "non-fundamental" solutions is open to question. More recently, the criterion of "expectational stability" or "learnability" has been proposed (Evans and Honkapohja 2001), and used by McCallum (2009) to select the level of inflation targeted by a Taylor rule, although Cochrane (2009) contests this result. As Subsection 4.2 shows, market selection offers another avenue for selection between multiple rational expectations equilibria.

B Monetary economy derivation

In this appendix, I provide a detailed derivation connecting the model of Section 2 and 3 with the example of Subsection 4.2. It is a version of the popular Woodford (2003, §2.1) complete markets cashless economy, modified to allow for consumer heterogeneity.

Let a *money plan* $m^i : \Sigma \to \prod_{t=0}^{\infty} \mathbb{R}_+$ and an *asset plan* $b^i : \Sigma \to \prod_{t=0}^{\infty} \mathbb{R}_+$ be sequences of $\mathscr{F}_t$-measurable $\mathbb{R}_+$-valued functions $\{m_t^i(\sigma)\}_{t=0}^{\infty}$ and $\{b_t^i(\sigma)\}_{t=0}^{\infty}$ capturing respectively consumer i's end-of-period nominal holdings of money (the economy's unit of account) and of all other financial assets. If $\varsigma_t^s(\sigma)$ is the cylinder set $\{\sigma^{t-1} \times s\} \times S \times S \times \cdots$, each consumer i faces the problem

$$\begin{aligned} &\max_{c^i, m^i, b^i} \mathrm{E}_{p^i} \left\{ \sum_{t=0}^{\infty} \beta_i^t u_\sigma^i(c_t^i(\sigma)) \right\}, \quad \text{s.t. } \forall t, \sigma^{t-1}, s: \\ &\gamma_t(s) c_t^i(\varsigma_t^s(\sigma)) + m_t^i(\varsigma_t^s(\sigma)) + b_t^i(\varsigma_t^s(\sigma)) \leq \gamma_t(s) \omega_t^i(\varsigma_t^s(\sigma)) + a_t^i(\varsigma_t^s(\sigma)); \qquad (8) \\ &c_t^i(\varsigma_t^s(\sigma)), m_t^i(\varsigma_t^s(\sigma)), b_t^i(\varsigma_t^s(\sigma)) \geq 0, \end{aligned}$$

where $\gamma_t(s)$ is the state-contingent price of the good in terms of the monetary unit, and $a_t^i(\varsigma_t^s(\sigma))$ is the total nominal value of all assets (including money) at the beginning of period t in state s (given σ^{t-1}).

Letting $n_{t+1}^i(\sigma)$ be the state-contingent value of nonmonetary assets in period $(t+1)$, the absence of arbitrage opportunities implies the existence of a (unique) nominal *stochastic discount factor* $D_{t,t+1}$ with the property that $b_t^i(\sigma) = \mathrm{E}_t[D_{t,t+1} n_{t+1}]$, where E_t denotes the conditional expectation $\mathrm{E}_p(\cdot|\sigma^t)$. Letting R_t^m be the *nominal interest*

[19] Transversality conditions rule out explosions in real variables by requiring their present value to converge to zero as time goes to infinity.

rate paid on money balances held at the end of period t, the beginning-of-period value of all assets is given by $a^i_{t+1}(\sigma) = (1 + R^m_t)m^i_t(\sigma) + n^i_{t+1}(\sigma)$, and the budget constraint (8) becomes

$$\begin{aligned} &\gamma_t(s)c^i_t(\varsigma^s_t(\sigma)) + m^i_t(\varsigma^s_t(\sigma)) + \mathrm{E}_t\left[D_{t,t+1}\left(a^i_{t+1} - (1 + R^m_t)m^i_t\right)\right] \\ &\qquad\qquad\qquad\qquad \leq \gamma_t(s)\omega^i_t(\varsigma^s_t(\sigma)) + a^i_t(\varsigma^s_t(\sigma)) \\ \Leftrightarrow\quad &\gamma_t(s)c^i_t(\varsigma^s_t(\sigma)) + \nu_t m^i_t(\varsigma^s_t(\sigma)) + \mathrm{E}_t\left[D_{t,t+1}a^i_{t+1}\right] \\ &\qquad\qquad\qquad\qquad \leq \gamma_t(s)\omega^i_t(\varsigma^s_t(\sigma)) + a^i_t(\varsigma^s_t(\sigma)), \end{aligned}$$

where $\nu_t \equiv (R_t - R^m_t)/(1 + R_t)$, and R_t is the riskless one-period *nominal interest rate* that solves $1/(1 + R_t) = \mathrm{E}_t\, D_{t,t+1}$ (i.e. the nominal interest rate on nonmonetary assets).[20] This constraint (and a borrowing limit ruling out Ponzi schemes) is satisfied in each period if and only if

$$\begin{aligned} &\sum_{t=0}^{\infty}\sum_{\sigma^{t-1}}\sum_{s\in S} p_t(\varsigma^s_t(\sigma))D_{0,t}(\varsigma^s_t(\sigma))\left[\gamma_t(s)c^i_t(\varsigma^s_t(\sigma)) + \nu_t m^i_t(\varsigma^s_t(\sigma))\right] \\ &\qquad\qquad \leq a^i_0(\emptyset) + \sum_{t=0}^{\infty}\sum_{\sigma^{t-1}}\sum_{s\in S} p_t(\varsigma^s_t(\sigma))D_{0,t}(\varsigma^s_t(\sigma))\gamma_t(s)\omega^i_t(\varsigma^s_t(\sigma)), \end{aligned}$$

where $\sigma^{-1} \equiv \emptyset$, p puts probability 1 on some initial state σ_0, and $D_{t,T} \equiv \prod_{\tau=t+1}^{T} D_{\tau-1,\tau}$; this is the single budget constraint characteristic of complete markets (see Woodford 2003, Proposition 2.1).

Together with the absence of arbitrage opportunities then, the first-order conditions for agent i's optimal choice of consumption imply that, for all t and σ:

1. there is a number $\alpha_i > 0$ such that if $p^i_t(\sigma) > 0$, then

$$\beta^t_i u^i_\sigma{}'(c^i_t(\sigma))p^i_t(\sigma) - \alpha_i p_t(\sigma)D_{0,t}(\sigma)\gamma_t(\sigma_t) = 0; \tag{9}$$

2. if $p^i_t(\sigma) = 0$, then $c^i_t(\sigma) = 0$.

[20] The latter is because $\mathrm{E}_t\, D_{t,t+1}$ prices the risk-free period-$(t + 1)$ asset; since the model has just one consumption good, this asset will simply pay 1 unit of consumption in each period-$(t+1)$ state. It is standard to set $\nu_t = 0$ (i.e. $R_t = R^m_t$) in equilibrium, so that positive quantities of money are held.

From (9) we know that, under equal discount factors, for all i, t and σ,

$$\frac{\beta u^{i\prime}(c^i_{t+1}(\sigma))p^i_{t+1}(\sigma)}{u^{i\prime}(c^i_t(\sigma))p^i_t(\sigma)} = \frac{p_{t+1}(\sigma)D_{0,t+1}(\sigma)\gamma_{t+1}(\sigma_{t+1})}{p_t(\sigma)D_{0,t}(\sigma)\gamma_t(\sigma_t)}$$

$$\sum_{s\in S}\frac{\beta u^{i\prime}(c^i_{t+1}(\varsigma^s_{t+1}(\sigma)))p^i_{t+1}(\varsigma^s_{t+1}(\sigma)|\sigma^t)}{u^{i\prime}(c^i_t(\sigma))} = \sum_{s\in S}\frac{p_{t+1}(\varsigma^s_{t+1}(\sigma)|\sigma^t)D_{t,t+1}(\varsigma^s_{t+1}(\sigma))\gamma_{t+1}(s)}{\gamma_t(\sigma_t)}$$

$$\mathrm{E}_t\, d_{t,t+1} = \mathrm{E}_t\, D_{t,t+1}\Pi_t, \tag{10}$$

where $\Pi_t \equiv \gamma_{t+1}/\gamma_t$ is the ratio of the price of consumption in period $t+1$ to that in period t and

$$d_{t,t+1}(\varsigma^s_{t+1}(\sigma)) = \frac{\beta u^{i\prime}(c^i_{t+1}(\sigma))}{u^{i\prime}(c^i_t(\sigma))}\frac{p^i_{t+1}(\varsigma^s_{t+1}(\sigma)|\sigma^t)}{p_{t+1}(\varsigma^s_{t+1}(\sigma)|\sigma^t)}$$

is the (unique) real stochastic discount factor prevailing in the cylinder $\varsigma^s_{t+1}(\sigma)$ under complete markets (Harrison and Kreps 1979, Sargent 2008 p. 9). Equation (10) is itself a form of "Fisher" (or frictionless IS) equation, but in the limit as p approaches a Dirac measure we have that

$$\frac{1}{1+r_t} = \pi^e_{t+1}\frac{1}{1+R_t},$$

where r_t is the real rate of interest and $\pi^e_{t+1} = \mathrm{E}_t(\Pi_t|\sigma^t)-1$ is the expected period-$(t+1)$ rate of inflation; linearization then gives a more familiar Fisher relation,

$$R_t = r_t + \pi^e_{t+1}. \tag{F}$$

Clearly the economy's path is undetermined as it stands, so I add a Taylor rule with a target rate of inflation π^*_t,

$$R_t = \pi^*_t + r_t + \frac{1}{a}(\pi_t - \pi^*_t), \tag{TR}$$

and I assume that there is a "Ricardian" fiscal policy (that adjusts to satisfy the government budget constraint at any prices).[21]

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[21]Note that I omit error terms from the model for simplicity, and thus deal with perfect foresight equilibria. For an analysis of the noisy case, see Cochrane (2009).

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