Week 2 - Preferences and Market Demand Curves

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The Cobb-Douglas utility function of the form $U = X^{\alpha}Y^{1-\alpha}$, where $0 < \alpha < 1$, is one of the most commonly used in economics. It has a number of neat mathematical properties, which we will derive below:

(a) The marginal utility of good X is found by differentiating the utility function with respect to X whilst holding Y constant. This yields $MU_X = \frac{\partial U}{\partial X} = \alpha X^{\alpha-1}Y^{1-\alpha}$. This rearranges to give: $MU_X = \alpha\left(\frac{Y}{X}\right)^{1-\alpha}$. By the same method, but this time differentiating with respect to Y and holding X constant, we get $MU_Y = \frac{\partial U}{\partial Y} = (1-\alpha)X^{\alpha}Y^{1-\alpha-1} = (1-\alpha)\left(\frac{X}{Y}\right)^{\alpha}$. Under Varian's definition, the marginal rate of substitution (MRS) is equal to the negative of the ratio of marginal utilities: $MRS_{X,Y} = -\frac{MU_X}{MU_Y}$. Plugging in the values for MU_X and MU_Y derived above gives us: $MRS_{X,Y} = -\left(\alpha\left(\frac{Y}{X}\right)^{1-\alpha}\right)\left((1-\alpha)\left(\frac{X}{Y}\right)^{\alpha}\right)^{-1} = -\frac{\alpha}{1-\alpha}\left(\frac{Y}{X}\right)$.

(b) We can see from these formulas that the marginal utility from good X is decreasing in the amount consumed of good X but increasing in the amount consumed of good Y, and that the marginal utility of good Y is decreasing in the amount consumed of good Y but increasing in the amount consumed of good X. If we are using cardinal utility (i.e. assuming that the amount as well as the order of utility for different bundles is meaningful) then this has a straightforward economic interpretation: As more of good X or Y is consumed, the consumer gains less additional utility from each extra unit. However, having more of good X increases the marginal utility of good Y, and vice versa, because the goods are to some degree complementary (as shown by the convexity of the indifference curves, which we will derive below) and so additional X makes Y more valuable at the margin, and vice versa.

(c) By setting $MRS_{X,Y} = -\frac{p_X}{p_Y}$ we get $-\frac{\alpha}{1-\alpha}\left(\frac{Y}{X}\right) = -\frac{p_X}{p_Y}$. This rearranges to give $\frac{p_Y}{p_X} = \frac{1-\alpha}{\alpha}\left(\frac{X}{Y}\right)$. Rearranging the budget constraint $M = Xp_X + Yp_Y$ gives us $\frac{M}{p_X} - X = Y\frac{p_Y}{p_X}$. Combining these two equations yields: $\frac{M}{p_X} - X = Y\left(\frac{1-\alpha}{\alpha}\frac{X}{Y}\right)$. This simplifies to: $\frac{M}{p_X} = \frac{1}{\alpha}X$. Making X the subject of this formula yields the equation for the Marshallian demand curve $X_D = \frac{\alpha M}{p_X}$. A similar method will yield the Marshallian demand for good Y $Y_D = \frac{(1-\alpha)M}{p_Y}$.

(d) The inverse Marshallian demand function expresses price as a function of quantity rather than quantity as a function of price. To derive it, we simply make price the subject of the above formula, yielding $p_X = \frac{\alpha M}{X_D}$. A number of features of the Marshallian demand curves produced from Cobb-Douglas preferences become immediately obvious. Firstly, there are no cross-price effects - the price of good Y does to affect the amount demanded of good X, and vice versa. As we shall see later on, this is because Cobb-Douglas preferences always result in a constant proportion of income being spent on each good. It follows directly from this that the Engel curves will be straight lines with a constant slope of $\frac{\partial X_D}{\partial M} = \frac{\alpha}{p_X}$ and $\frac{\partial Y_D}{\partial M} = \frac{1-\alpha}{p_Y}$ respectively. The income offer curves are therefore also straight lines through the origin. This is the geometric property satisfied by homothetic preferences, of which Cobb-Douglas preferences are one example (along with perfect complements and perfect substitutes). It can also be shown algebraically that Cobb-Douglas preferences are homothetic because if $X^\alpha Y^{1-\alpha} > (X')^\alpha (Y')^{1-\alpha}$ then $(tX)^\alpha (tY)^{1-\alpha} > (tX')^\alpha (tY')^{1-\alpha}$. (This is ensured mathematically because the Cobb Douglass utility function is homogeneous, meaning that $U(tX, tY) = t^N U(X, Y)$ where N is the degree of homogeneity.)

(e) There are two ways to derive a formula for the MRS that we can then differentiate with respect to X in order to show that the absolute value is declining, and therefore that the indifference curves are convex. If we take our earlier formula $MRS_{X,Y} = -\frac{\alpha}{1-\alpha}\left(\frac{Y}{X}\right)$, the problem is that we cannot differentiate this with respect to X because along an indifference curve Y is not constant with respect to X (it must decrease as X increases in order to keep utility constant). By taking a constant utility of $\bar{U}$ and rearranging the utility function to give $Y = \left(\frac{\bar{U}}{X^\alpha}\right)^{\frac{1}{1-\alpha}}$ (this gives us the equation for the indifference curve where $U = \bar{U}$), we can substitute this in to the MRS equation so that the MRS is expressed entirely in terms of X and $\bar{U}$, which is a constant with respect to X. This gives us: $MRS_{X,Y} = -\frac{\alpha}{1-\alpha}\left(\frac{\left(\frac{\bar{U}}{X^\alpha}\right)^{\frac{1}{1-\alpha}}}{X}\right)$, which simplifies to give $MRS_{X,Y} = -\frac{\alpha}{1-\alpha}\bar{U}^{\frac{1}{1-\alpha}}X^{-\frac{\alpha}{1-\alpha}-1}$ and further to give $MRS_{X,Y} = -\frac{\alpha}{1-\alpha}\left(\frac{\bar{U}}{X}\right)^{\frac{1}{1-\alpha}}$. Alternatively, we could have taken the formula for the indifference curve, and differentiated it to give $\frac{\partial}{\partial X}\left(\left(\frac{\bar{U}}{X^{\alpha}}\right)^{\frac{1}{1-\alpha}}\right) = \frac{\partial}{\partial X}\left(\bar{U}^{\frac{1}{1-\alpha}}X^{-\frac{\alpha}{1-\alpha}}\right) = -\frac{\alpha}{1-\alpha}\bar{U}^{\frac{1}{1-\alpha}}X^{-\frac{\alpha}{1-\alpha}-1} = -\frac{\alpha}{1-\alpha}\left(\frac{\bar{U}}{X}\right)^{\frac{1}{1-\alpha}}$. Since the MRS is the derivative (gradient) of the indifference curve, this method gives the same result.

(f) In order to show that the indifference curves are convex we must find the derivative of the expression for the MRS, which is the second derivative of the expression for the value of y along an indifference curve in terms of x. This gives us $\frac{\partial}{\partial X}\left(-\frac{\alpha}{1-\alpha}\left(\frac{\bar{U}}{X}\right)^{\frac{1}{1-\alpha}}\right) = \frac{\partial}{\partial X}\left(-\frac{\alpha}{1-\alpha}\bar{U}^{\frac{1}{1-\alpha}}X^{-\frac{1}{1-\alpha}}\right) = \left(-\frac{\alpha}{1-\alpha}\right)\left(-\frac{1}{1-\alpha}\right)\bar{U}^{\frac{1}{1-\alpha}}X^{-\frac{1}{1-\alpha}-1} = \left(\frac{\alpha}{(1-\alpha)^2}\right)\bar{U}^{\frac{1}{1-\alpha}}X^{-\frac{2-\alpha}{1-\alpha}} = \left(\frac{\alpha}{(1-\alpha)^2}\right)\left(\frac{\bar{U}}{X^{2-\alpha}}\right)^{\frac{1}{1-\alpha}}$. This derivative can be seen to be unambiguously positive, showing that the indifference curves are downward sloping but with an increasingly shallow gradient, and are therefore convex.

(g) From the above parts, we can see that the consumer's preferences are consistent and well-behaved. Firstly, since the utility function assigns a unique real number to every possible bundle, this ensures that preferences are complete, reflexive and transitive, because real numbers have these properties. Secondly, because MU_X and MU_Y are always positive, this shows that monotonicity (or non-satiation - they mean the same thing) will be satisfied. This is also reflected in the fact that the expression for the MRS is always negative, showing that the indifference curves are downward-sloping. Finally, the answer to part (f) shows that the consumer's preferences satisfy convexity, because the indifference curves are convex.

(h) When we add together the demand of N identical consumers, the aggregate demand curve is given by $X_{DA} = NX_D = N\frac{\alpha M}{p_X}$. When we have N consumers with different income levels M and values of α in their utility functions, we get $X_{DA} = \sum_{i=1}^{N} X_{Di} = \sum_{i=1}^{N}\left(\frac{\alpha_i M_i}{p_X}\right) = \frac{\sum_{i=1}^{N}(\alpha_i M_i)}{p_X} = N\frac{\sum_{i=1}^{N}\left(\frac{\alpha_i M_i}{N}\right)}{p_X}$. From this form, we can see that the aggregate demand curve is the same as N multiplied by the demand of the representative consumer, for whom αM is the average of all the consumers $\sum_{i=1}^{N}\left(\frac{\alpha_i M_i}{N}\right)$.

(i) The total amount, or revenue, spent on good X by the aggregated consumers will be $R_X = (X_{DA})(p_X)$. Plugging in the expression for the Marshallian demand curve gives us $R_X = \sum_{i=1}^{N}(\alpha_i M_i)$. The total amount spend on good X depends only on the consumers' incomes M_i and the α_i parameters. This is due to the particular special property of the Cobb-Douglas utility function that a fixed proportion of income is spent on each good. When the price of a good goes up, the amount demanded goes down just sufficiently to keep the total amount spent constant. The reverse occurs when the price of a good goes down.

(j) By letting the parameter B equal $\sum_{i=1}^{N}(\alpha_i M_i)$, or $NM\alpha$ if all individuals are identical, we can express the Marshallian demand function as $X_D = \frac{B}{p_X}$. From the form, it is clear that aggregating together any number of individuals with Cobb-Douglas preferences leads to a demand function of the same form as a single individual. Analysis of the role of income and substitution effects from the aggregate demand curve will apply to all individuals similarly. For example, the analysis in question 2 showing that both goods must be normal will apply to all the individuals with Cobb-Douglas preferences who are added together into a particular aggregate demand curve. In the real world, however, there is no reason to assume that all individuals have the same kind of preferences. Just because a good observed to be normal from the aggregate Engel curve, for instance, will not in general imply that is is normal for all individuals. It may be inferior to some but sufficiently normal to others to outweigh this so that this cannot be observed at the aggregate level.

(l) The first derivative of the expression for the Marshallian demand curve with respect to price p_X is $\frac{\partial}{\partial p_X}\left(\frac{B}{p_X}\right) = -\frac{B}{(p_X)^2}$. The second derivative is $\frac{\partial^2}{\partial {p_X}^2}\left(\frac{B}{p_X}\right) = 2\frac{B}{(p_X)^3}$. Since the first derivative is negative and the second is positive we know that the Marshallian demand curve is downward sloping and convex.

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See graph at the end.

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(a) Taking the quasilinear utility function $U = X^\alpha + Y$, where $0 < \alpha < 1$, and assuming that Y is the numeraire good so that $p_Y = 1$, the $MRS_{X,Y} = -\frac{MU_X}{MU_Y} = -\frac{p_X}{p_Y}$ condition simplifies to give $-\alpha X^{\alpha-1} = -p_X$ (because $MU_Y = 1$). This rearranges to give $X_D = \left(\frac{\alpha}{p_X}\right)^{\frac{1}{1-\alpha}}$.

(b) Demand does not depend upon income because with quasilinear preferences all indifference curves have the same gradient at any particular value of X (they are simply vertical translations of each other). This means that the income effect only changes demand for good Y (once we have income high enough to ensure an interior optimum).

(c) The aggregate Marshallian demand curve for N identical consumers will be $X_{DA} = NX_D = N\left(\frac{\alpha}{p_X}\right)^{\frac{1}{1-\alpha}} = \left(\frac{N^{1-\alpha}\alpha}{p_X}\right)^{\frac{1}{1-\alpha}}$.